STOCHASTIC MODELS FOR STRUCTURE OF BREEDING DAIRY POPULATION*

BY

J. P. JAIN

I.A.S.R.I., New Delhi

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SUMMARY

Two stochastic analogues of the deterministic discrete model (Jain and Narain [4]) for studying the growth of bisexual dairy population grouped in unequal stage-groups have been presented. The number of individuals born during any arbitrary interval of time is assumed to follow a Poisson/binomial distribution and the number of transfers occurring during the same interval a binomial distribution. Linear matrix recurrence relation is derived which determines precisely the first two moments of the stage-group random variables at each unit of time. Further, Monte Carlo experiments using the two models are described and their results compared.

I. INTRODUCTION

Within the framework of optimum breeding plans, for a given schedule of scheme of selection and vital characteristics, is the study of growth and structure of the breeding population. For studying the pattern of growth of dairy *female* population grouped in unequal stage-groups Jain and Narain [3] adapted Lefkovitch's [5] deterministic model. Subsequently, Jain and Narain [4] extended the model to include both the sexes. In both these deterministic formulations time and age-scale variables were treated as discrete and the number of individuals in various stage-groups as fixed. However, when the fact that the population considered is not infinite cannot be ignored it would be more appropriate to consider the probabilistic approach as this takes into account the random fluctuation which arises

* This paper is in continuation of earlier papers of Jain and Narain (1974, 1979) and Jain (1977). The reader is therefore referred to first read those papers so as to acquaint himself with the methodology, concepts, notations and results used in this paper.

as a result of finiteness of the population. Jain [2] gave two stochastic analogues of the deterministic model of Jain and Narain [3] for dairy female population. The number of individuals born during any arbitrary interval of time was assumed to follow either a Poisson distribution or a binomial distribution and the number of transfers occurring during the same interval a binomial distribution. The two models were referred as Poisson-binomial model and Binomial-binomial model. In the present paper the same two stochastic models are extended to cover the bisexual set up of the population, thus providing the stochastic versions of the two-sex deterministic model of Jain and Narain [4]. Linear matrix recurrence relations are derived for the means, variance and covariance of the random numbers in the various stage-groups.

2. The Stochastic Models

2.1. The deterministic model :

In a bisexual set up, we consider the male and female population at discrete intervals of time grouped respectively in r and s stages in the manner described in Jain and Narain [4]. If $\underline{m}_t$ and $\underline{n}_t$ are the respective male and female population vectors at time t, then the composition of the population at time $t+1$ can be obtained deterministically by the relation

$$\begin{bmatrix} \underline{m}_{t+1} \\ \underline{n}_{t+1} \end{bmatrix} = H \begin{bmatrix} \underline{m}_t \\ \underline{n}_t \end{bmatrix} \tag{1}$$

where H is a $(r+s)\times(r+s)$ matrix embodying the regime of mortality, fertility and culling rates supposed to apply over the interval, and is of the form

$$H = \begin{bmatrix} T & B \\ O & F \end{bmatrix} \tag{2}$$

in which $T\,[=((c_{ij}))]$ is $r\times r$ for males and by the row-by-column multiplication rule T will transfer the male stock into the succeeding stage-groups, $B\,[=((b_{ij}))]$ is $(r\times s)$ also for males and the elements of which when multiplied by the column vector of females will give the number of males entering a particular stage-group through births;

and $F[=((a_{ij}))]$ is $s \times s$ for females. The non-zero elements of T, B and F are

T : c_{31} and $c_{i+1,i}$ $(i=1,2,\ldots, r-1)$ B : b_{ij} $(i=1$ and 2, and $j=4,5,\ldots,s)$	Jain and Narain [4]
F : a_{ij} $(i=1$ and 2, and $j=4,5,\ldots,s)$ a_{31} and a_{ij} $(i=4$ and $j=1,2$ and $4)$ $a_{i+1,i}$ $(i=1,2, \ldots, s\text{-}1)$	Jain and Narain [3]

2.2. Parameters of the stochastic models :

In addition to the parameters introduced in the earlier paper (Jain [2]), we assume that the number of males in stage group y at time t is a random variable m_{yt} with expected value θ_{yt} and variance $\gamma^{(t)}_{y,y}$ and that covariance (m_{yt}, m'_{yt}) $= \gamma^{(t)}_{y,y'}$. Further, let covariance $(m_{yt}, n_{xt}) = \beta^{(t)}_{y,x}$, where n_{xt} is the number of females in stage group x at time t assumed a random variable with expectation e_{xt}, variance $C^{(t)}_{x,x}$ and covariance $(n_{xt}, n'_{xt}) = C^{(t)}_{x,x'}$. It is to be noted that for $i \neq j$, $\beta^{(t)}_{i,j} \neq \beta^{(t)}_{j,i}$. We also denote $h_{iy} = 1 - c_{iy}$.

2.3 Poisson-binomial model:

Consider the n_{xt} females in stage-group x at time t. Each of them has fixed probabilities b_{ix} of contributing a male to stage group i $(< x)$ at time $t+1$. That is to say n_{xt} mothers contribute sons to stage group i $(<x)$ at the mean rate of $n_{xt}b_{ix}$ per unit of time.

Let, for $i<x, m_{i,t+1}^{(x)}$ denote the number of males contributed and remaining at time $t+1$ in stage group i by the mothers who were in stage group x at time t ; and for $i \geqslant y$, $m_{i,t+1}^{(y)}$ denote the number of males in stage group i at time $t+1$ transferred from among those who were in stage group y at time t. From the same considerations of Section (2.3) of the earlier paper (Jain [2]), for $i<x$, $m_{i,t+1}^{(x)}$ can be assumed a Poisson variable $P\,(n_{xt}b_{ix})$ conditional on n_{xt} and for $i \geqslant y$, $m_{i,t+1}^{(y)}$ a binomial variable $B\,(m_{yt}, c_{iy})$ conditional on m_{yt}.

With the foregoing formulation of the stochastic model and using the expressions (4) to (11) of the previous paper (Jain [2]) the means, variances and covariances of the number of males and females in different stage groups can be easily obtained (Jain [1]). If we denote

$$\underline{\alpha}_i^{(t)} = \begin{bmatrix} \underline{\gamma}_i^{(t)} \\ \\ \underline{\beta}_i^{(t)} \end{bmatrix}; \quad \underline{\delta}_i^{(t)} = \begin{bmatrix} \underline{\beta}_i^{(t)} \\ \\ \underline{C}_i^{(t)} \end{bmatrix}; \quad \underline{K}_t = \begin{bmatrix} \underline{\theta}_t \\ \\ \underline{e}_t \end{bmatrix}$$

and

$$\underline{D}'^{(t)}=\left[\ \underline{\alpha}_1^{(t)}\ \ \underline{\alpha}_2^{(t)}\ \cdots\cdots\ \underline{\alpha}_r^{(t)}\ \ \underline{\delta}_1^{(t)}\ \ \underline{\delta}_2^{(t)}\cdots\ \underline{\delta}_s^{(t)}\ \right]$$

where

$$\underline{\gamma}'^{(t)}_{i}=\left[\ \gamma_{i1}^{(t)}\ \ \gamma_{i2}^{(t)}\cdots\cdots\ \gamma_{ir}^{(t)}\ \right];$$

$$\underline{\beta}'^{(t)}_{i}=\left[\ \beta_{i1}^{(t)}\ \ \beta_{i2}^{(t)}\cdots\cdots\ \beta_{is}^{(t)}\ \right];$$

$$\underline{\beta}'^{(t)}_{\cdot i}=\left[\ \beta_{1i}^{(t)}\ \ \beta_{2i}^{(t)}\cdots\cdots\ \beta_{ri}^{(t)}\ \right];$$

and

$$\underline{C}'^{(t)}_{i}=\left[\ C_{i1}^{(t)}\ \ C_{i2}^{(t)}\cdots\cdots\ C_{is}^{(t)}\ \right]$$

then the first two moments in the bisexual set up can be shown to be given by the following recurrence relation :

$$\begin{bmatrix} \underline{K}_{t+1} \\ \underline{D}^{(t+1)} \end{bmatrix}=\begin{bmatrix} H & 0 \\ G & H\times H \end{bmatrix}\begin{bmatrix} \underline{K}_t \\ \underline{D}^{(t)} \end{bmatrix} \quad \text{..(3)}$$

where G is $(r+s)^2\times(r+s)$ which can be decomposed into $(r+s)$ submatrices as follows

$$G'=[G_1\ G_2\ \cdots\cdots\ G_{r+s}] \quad \text{...(4)}$$

in which each G_i is $(r+s)\times(r+s)$, the elements g_{ij} being the (i, j)th element of H multiplied by its complement excepting when $i<j$ for which g_{ij} equals the (i, j)th element of H; and the elements of all other rows being zero.

Recursing relation (10), we get

$$\begin{bmatrix} \underline{K}_t \\ \underline{D}^{(t)} \end{bmatrix}=\begin{bmatrix} H & 0 \\ G & H\times H \end{bmatrix}^t\begin{bmatrix} \underline{K}_0 \\ \underline{D}^{(0)} \end{bmatrix}$$

$$=\begin{bmatrix} \underline{K}_t \\ (H\times H)^t\,\underline{D}^{(0)}+\sum_{j=1}^{t}(H\times H)^{t-j}\,G\,\underline{K}_{j-1} \end{bmatrix} \quad \text{...(5)}$$

These relations are exactly of the same form as obtained earlier for unisex population (Jain [2]). The asymptotic behaviour of the population can thus be dealt with in much the same manner as for unisex population discussed earlier.

2.4 Binomial-binomial model :

Following Pollard [6] in addition to viewing the number of transfers $m^{(y)}_{i,t+1}$ from stage-group y to i ($i \geqslant y$) during the time interval $(t, t+1)$ as a binomial variable, the number of births taking place during the same interval *i.e.* $m^{(x)}_{i,t+1}$ ($i < x$) can also be considered as a binomial variate $B(n_{x_t}, b_{ix})$ conditional on n_{x_t}. With this formulation, the first two moments still are given by the relations (3) and (5) excepting that the element g_{ij} of matrix G_i in relation (4) is equal to the (i, j)th element of H multiplied by its complement for all i.

3. Monte Carlo Experiments Using the Two Models

Let for a dairy cattle herd the values of non-zero elements of different matrices be as shown in Table 1. In this herd the male

TABLE 1

Values of non-zero elements of the matrices

Matrix	*Values of elements*		
T	$c_{21}=0.0334$	$c_{32}=0.9268$	$c_{65}=0.9875$
	$c_{31}=0.0338$	$c_{43}=0.9875$	$c_{76}=0.9875$
		$c_{54}=0.9052$	$c_{87}=0.2963$
B	$c_{14}=0.2417$	$b_{24}=0.0042$	
	$b_{1i}=0.3062$ ($i \geqslant 5$)	$b_{2i}=0.0054$ ($i \geqslant 5$)	
F	$a_{14}=0.2417$	$a_{2i}=0.0731$ ($i \geqslant 5$)	$a_{44}=0.1723$
	$a_{1i}=0.3061$ ($i \geqslant 5$)	$a_{31}=0.2279$	$a_{64}=0.5954$
	$a_{21}=0.6896$	$a_{42}=0.8591$	$a_{i+1,i}=0.7876$ ($i=5$ to 7)
	$a_{24}=0.0577$	$a_{43}=0.8004$	$a_{98}=0.8423$

stock is assumed to be grouped in eight distinct stage groups and the female stock into nine stage-groups. Suppose the female strength of this herd at some stage of the progeny testing programme in which 10 bulls per set comprising two tested and eight under test are in use at a time, consists of about 300 adult females alongwith their followers classified in different stage groups as follows :

$$\underline{n}'_o = [80 \quad 75 \quad 25 \quad 125 \quad 55 \quad 40 \quad 35 \quad 25 \quad 20]$$

Let the corresponding vector of male stock has its elements as

$$\underline{m}'_o = [80 \quad 12 \quad 11 \quad 10 \quad 9 \quad 9 \quad 9 \quad 3]$$

Since constancy of the female herd strength is one of the requirements of the programme, the present example has been constructed to meet this requirement by suitable choice of controlling factors $s_1, s_2, \ldots$ described in the earlier paper (Jain and Narain, [4]). This can be seen from the fact that the dominant latent root of matrix F is unity.

The unit of time taken is one calving interval which for this herd has been taken as 15 months,

Table 2 gives the expected numbers and variances in different stage groups as also the simulated numbers for both Poisson-binomial and binomial-binomial models at two consecutive points of time *viz.* at 20 and 21 units of time. The expected numbers and their variances are theoretical values for the means and variances obtained using the iterative equation (3) with appropriate values of the elements of matrix G under the respective two models.

The variances of the numbers in different stage-groups under both the models are more or less of the same order in both the sexes. Further, the standard normal deviation of the actual numbers from the theoretical means at two consecutive points of time under the Poisson-binomial model ranged from -1.41 to 2.12 in males and -1.54 to 0.45 in females as against -2.27 to 0.45 and -1.40 to 0.60 under the other model. These considerations imply that either of the two models can be used in the study of growth in dairy populations.

4. Summing Up

Even though the stochastic models formulated in this paper are for dairy population breeding under optimum mating and selection schemes, the same with suitable modifications of the elements of the generation matrix H can be used for most uniparous species such as cattle, buffaloes and birds. These are also good approximations for organisms, such as man, where the population, although changing continuously, is censused at discrete intervals.

TABLE 2

Monte Carlo experimental results using the two models

Stage-group	*Poisson—binomial model*				*Binomial—binomial model*			
	ctual	*Expected*	*Variance*	*Norm. devn.**	*Actual*	*Expected*	*Variance*	*Norm. devn.**
1	2	3	4	5	6	7	8	9
				(a) *After 20 units of time*				
Male								
1	70	86	231	—1.05	83	86	194	—0.21
2	1	4	5	—1.34	5	4	5	0.45
3	7	7	8	0.00	8	7	8	0.35
4	8	7	8	0.35	5	7	8	—0.71
5	5	6	7	—0.38	5	6	7	—0.38
6	5	6	7	—0.38	7	6	7	0.38
7	11	6	7	1.89	0	6	7	—2.27
8	0	2	2	—1.41	2	2	2	0.00
Female								
1	67	86	231	—1.25	73	86	194	—0.93
2	71	80	198	—0.64	76	80	175	—0.30
3	12	20	27	—1.54	23	20	25	0.60
4	92	102	300	—0.58	88	102	268	—0.91
5	49	61	127	—1.06	54	61	116	—0.64
6	47	48	87	—0.11	46	48	80	—0.22
7	40	38	61	0.26	35	38	56	—0.40
8	20	30	43	—1.52	26	30	41	—0.62
9	25	25	34	0.00	22	25	32	—0.53

1	2	3	4	5	6	7	8	9
				(b) *After 21 units of time*				
Male								
1	74	86	238	—0.91	74	86	200	—0.85
2	4	4	5	0.00	4	4	5	0.00
3	4	7	8	—1.06	5	7	8	—0.71
4	8	7	8	0.35	8	7	8	0.35
5	8	6	7	0.76	4	6	7	—0.76
6	5	6	7	—0.38	4	6	7	—0.76
7	5	6	7	—0.38	7	6	7	0.38
8	5	2	2	2.12	0	2	2	—1.41
Female								
1	74	86	238	—0.91	75	86	200	—0.78
2	60	80	204	—1.40	69	80	180	—0.82
3	14	20	27	—1.15	13	20	25	—1.40
4	83	102	309	—1.08	99	102	277	—0.18
5	55	61	131	—0.52	54	61	120	—0.64
6	36	48	89	—1.27	43	48	82	—0.55
7	37	38	62	—0.13	37	38	58	—0.13
8	33	30	44	0.45	25	30	41	—0.78
9	17	25	34	—1.37	19	25	33	—1.04

*Standardised deviation of the simulated values from the theoretical mean.

As noted in the earlier paper (Jain and Narain [4]) the growth of population depends upon the value of the dominant latent root (ω_f) of matrix H (which is also the dominant root of matrix F). If it is greater than 1, the total population will increase at the rate of ω_f per unit of time. Since a population cannot grow for ever the models are obviously unrealistic for any long period of time. The growth rate will eventually be limited by all the factors that collectively make up the carrying capacity of the environment. Consequently, if these models are to be used for population projections the elements of the generation matrix should incorporate a suitable functional relationship of vital characteristics with density of the population.

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